№ 1 (3), 2021 ISSN: 2682 – 9517 (print) ISSN: 2683 – 0930 (online) INSTITUTE OF INFORMATICS AND INNOVATIVE TECHNOLOGIES

A NOTE ON A THERMAL IMPEDANCE OF A PLATE WITH PERIODIC BOUNDARY CONDITIONS:
A case relevant to house thermal insulation

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Abstract: — **A semi-infinite one-dimensional heat conduction problem demonstrating the feasibility the fractional calculus transient thermal impedance to be determined when periodic boundary conditions imposed at the interface ($x=0$). This is an academic problem relevant to external house thermal insulation surface layer exposed to daily - night sunlight heating. The method is purely analytic base and based on fractional semi-derivatives and semi-integrals.**

Keywords: — **external surface heating; fractional semi derivatives, thermal impedance.**

I. Introduction

I.1. Heat transfer model

A thermal source heating a semi-infinite medium is a general when high temperature heat sources interact with materials and this is a very old subject [1,2], but remaining scientifically important [3,4]. In general, the energy transport by heat in a semi-infinite material heated by a line (plane) isothermal source at $x=0$ is modelled by

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}\ ,\ T=T(x,t) \tag{1}$$

with boundary and initial conditions

$$-\lambda\,\partial T/\partial x\left.\frac{\partial T}{\partial x}\right|_{x=0}=q_s, x=0,\ t>0 \tag{2a}$$

$$T(\infty,t)=T_\infty,\quad T(x,0)=T_\infty,\ t=0 \tag{2b}$$

The thermal impedance of the semi-infinite medium is defined as the ratio

$$Z=\left(T_s-T_\infty\right)/q_s \tag{3a}$$

This in the classical solution with the Gaussian error function for the entire spatial domain we have

$$Z_1=\left(2\sqrt{\alpha t}\right)/\left(\lambda\sqrt{\pi}\right). \tag{3b}$$

In order to clarify the problem at issue, if the energy equation (1) is governed by a prescribed temperature boundary condition

$$T(0,t)=T_s,\ t>0;\ T=T_\infty,\ t>0. \tag{4a}$$

Then, the exact solution [1] we have

$$\frac{T(x,t)-T_\infty}{T_s-T_\infty}=1-erf\left(x/2\sqrt{\alpha t}\right), \tag{4b}$$

and consequently the surface flux is

$$q_s=\frac{\lambda\left(T_s-T_\infty\right)}{\sqrt{\pi}\sqrt{\alpha t}}. \tag{4c}$$

Then,

from the definition of the thermal impedance we have

$$Z_2=\frac{\lambda\sqrt{\pi}}{\sqrt{\alpha t}}. \tag{5}$$

This simple introductory example demonstrates the theoretical background of the problem considered in this research note. The main problem emerging is how to avoid the entire domain $T(x,t)$ where the surface temperature and flux can be obtained by setting $x=0$.

1.2. Motivation of this study

The common approach in solution of heat transient problems is to develop complete solution over the entire domain $T(x,t)$, $0<x<L$, where L is the macroscopic length scale (the dimension of the body). Then, after cumbersome expressions of the solutions (see many examples in [1]) the step leading to definition of the thermal impedance requires to set $x=0$ it the solution to obtain $T(0,t)$ or $q_s=q(0,t)$, depending on the boundary conditions imposed.

However, there is an alternative approach where such cumbersome solutions can be avoided. This approach involve time-fractional derivatives [5,6,7] of order $1/2$.

This note addresses the thermal impedance of a semi-infinite plate heated by a periodic time dependent boundary condition. The solution is based on application of time-fractional semi-derivatives and integrals [6,7] briefly presented next.

I.3. Fractional-time derivative and Integrals: necessary information

The temperature and the flux governed by the model (1) are related by [6]

$$q(x,t)=\left(\lambda/\sqrt{\alpha}\right)\left[D_t^{1/2}T(x,t)-T_\infty/\sqrt{\pi t}\right] \qquad \text{(6a)}$$

And

$$T(x,t)=\left(\sqrt{\alpha}/\lambda\right)D_t^{-1/2}q(x,t)+T_\infty . \qquad \text{(6b}$$

The operators $\partial^{1/2}/t^{1/2}$ and $\partial^{-1/2}/t^{-1/2}$ are half-time fractional derivative and integral in the Riemann-Liouville sense [7], namely

$$\frac{\partial^{1/2}T}{\partial t^{1/2}}=\frac{1}{\Gamma(1/2)}\frac{d}{dt}\int_0^u\frac{T(x,t)}{\sqrt{t-u}}du-\frac{T(x,0)}{\sqrt{\pi t}}. \qquad (7)$$

Because the time fractional equivalent of the model (1) is [6,7]

$$\frac{D^{1/2}T(x,t)}{\partial t^{1/2}}=-\sqrt{a}\frac{\partial T(x,t)}{\partial x} \qquad (8)$$

and at $x=0$ we get

$$\frac{D^{1/2}T(0,t)}{\partial t^{1/2}}=-\sqrt{a}\frac{\partial T(0,t)}{\partial x}. \qquad (9)$$

Thus the surface flux is

$$q_s=-\lambda\frac{\partial T(0,t)}{\partial x}=\frac{\lambda}{\sqrt{a}}\frac{D^{1/2}T(0,t)}{\partial t^{1/2}}. \qquad (10)$$

Therefore, both the surface temperature and flux are simultaneously defined and related by the time-fractional semi derivative and semi-integral not only over the entire domain, but especially at the boundary $x=0$. The feasibility of these relationships will demonstrated by the examples solved next.

II. Periodic time-dependent boundary condition $x = 0$

II.1. Example 1: Periodic surface temperature

With a periodic temperature imposed at the surface $x=0$

$$T_s(t)=T_\infty+T_A\sin(\omega t) \qquad (11)$$

the boundary flux is [7]:

$$q_s(t)=\frac{\lambda}{\sqrt{\alpha}}D_t^{1/2}\left[T_0+T_A\sin(\omega t)\right]-\frac{\lambda}{\sqrt{\alpha}}\frac{T_\infty}{\sqrt{\pi t}} \qquad \text{(12a)}$$

$$q_s(t)=\frac{\lambda}{\sqrt{\alpha}}T_A\sqrt{\omega}\left\{\begin{matrix}\sin\left(\omega t+\frac{\pi}{4}\right)-\\-\sqrt{2}\Lambda\left(\sqrt{\frac{2\omega t}{\pi}}\right)\end{matrix}\right\} \quad (12b)$$

Thus, the surface thermal impedance is

$$Z_{T\sin(t)}=\frac{\sqrt{\alpha t}}{\lambda}\frac{\sin(\omega t)}{\sqrt{\omega t}\left\{\sin(\omega t+\pi/4)-\sqrt{2}\Lambda\left(\sqrt{2\omega t/\pi}\right)\right\}} \quad (13)$$

where Λ is the Fresnel function [6,7] (see comments below).

The Fresnel function Λ [7,8] in (13) is important for the initial stage of the transient but for long times we may assume $\Lambda(\tau)\approx 0$ and therefore $t\approx 10\pi/\omega$ when the condition $\max\left|\Lambda\left(\sqrt{2\omega t/\pi}\right)\right|<10^{-3}$ is obeyed. In such a case the we attain a steady-periodic regime [6]. Therefore, the steady-periodic heat flux is

$$q_s(t)=\left(\lambda/\sqrt{\alpha}\right)T_A\sqrt{\omega/2}\left[\sin(\omega t)+\cos(\omega t)\right]. \quad (14)$$

Then, the steady-periodic thermal impedance $\bar{Z}_{T\sin(t)}(t\to\infty)$ is

$$\bar{Z}_{T\sin(t)}(t\to\infty)=\frac{\sqrt{\alpha t}}{\lambda}\frac{\sin(\omega t)}{\sqrt{\omega t/2}\left[\sin(\omega t)+\cos(\omega t)\right]} \quad (15)$$

II.2. Example 1: Periodic surface flux

With a periodic surface flux

$$q_s(t)=q_A\sin(\omega t)$$

we have

$$T_s(t)=\frac{\sqrt{\alpha}}{\lambda}D_t^{-1/2}\left[q_A\sin(\omega t)\right]=$$
$$=\frac{\sqrt{\alpha}}{\lambda}q_A\sqrt{\omega}\left\{\begin{matrix}\sin\left(\omega t-\frac{\pi}{4}\right)+\\+\sqrt{2}\Omega\left(\sqrt{\frac{2\omega t}{\pi}}\right)\end{matrix}\right\} \quad (16)$$

$$Z_{q\sin(t)}=\frac{\sqrt{\alpha}}{\lambda}\frac{\sqrt{\omega}\left\{\begin{matrix}\sin(\omega t-\pi/4)+\\\sqrt{2}\Omega\left(\sqrt{2\omega t/\pi}\right)\end{matrix}\right\}}{\sin(\omega t)}-$$
$$-\frac{T_\infty}{q_A\sin(\omega t)} \quad (17a)$$

Similarly, as commented in Example 1 the Fresnel function Λ [7,8] vanishes at $t\approx 40\pi/\omega$, i.e. when the condition $\max\left|\Omega\left(\sqrt{2\omega/\pi}\right)\right|<4.10^{-2}$ [7] is obeyed.

In such a case the corresponding steady-periodic thermal impedance is

$$\bar{Z}_{1-q\sin(t)}=\frac{\sqrt{\alpha t}}{\lambda}\sqrt{\omega t}\left[\frac{\sin(\omega t-\pi/4)}{\sin(\omega t)}\right]-$$
$$-\frac{T_\infty}{q_A\sin(\omega t)} \quad (17b)$$

CONCLUSIONS

This note demonstrates how transient thermal impedances of semi-infinite plate with imposed periodic time-depending boundary conditions can be estimated by application of Riemann-Liouville fractional derivatives and integrals of order $1/2$.

The final formulae are especially re-casted in terms with pre-factor $\sqrt{\alpha t}/\lambda$ with dimension of thermal *impendence* $\left[K/Wm^2\right]$, thus outlining the effect of the material properties. The results are natural of the Fourier law leading to model (1), since when a semi-infinite medium is at issue the length scale is $\sqrt{\alpha t}$ because a natural one does not exist.

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